§1.6 Quantum Noise

Interactions between our quantum system and the environment $\longrightarrow$ decoherence

denote S = system, U = unitary op.,
E = environment

$\rightarrow$ reduced density matrix ρ_S':

$$\rho_S' = \mathrm{Tr}_E\left[U(\rho_S \otimes \rho_E)U^\dagger\right]$$

Using $\rho_E = \sum_k p_k |e_k\rangle\langle e_k|$, one obtains

$$\rho_S' = \sum_{(k',k)} K_{(k',k)}\, \rho_S\, K^\dagger_{(k',k)}$$

where

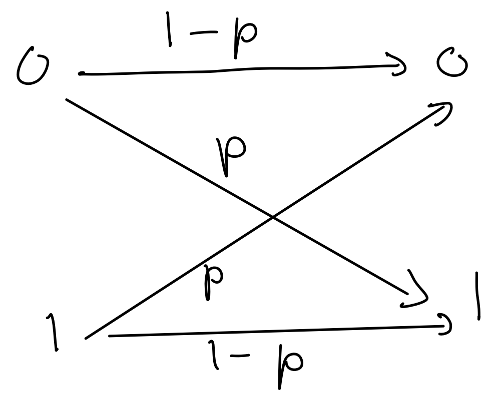
$$K_{(k,k')} = \sqrt{p_k}\, \langle e_k'| U |e_k\rangle$$

In general, a map of a quantum state is given as a completely-positive-trace-preserving (CPTP) map $\mathcal{E}$, which can be always written as:

$$\mathcal{E}\rho = \sum_{k=1}^{M} K_k \rho K_k^\dagger$$

In order to understand the properties of the operator $\mathcal{E}$, it is useful to think of a classical analogy:

imagine a bit stored on a hard disk
$\rightarrow$ interaction with external magnetic fields leads to flips over time:



where p is the prob. for a flip

Mathematically, we have

$$p(Y=y) = \sum_x \underbrace{p(Y=y \mid X=x)}_{\text{transition probability}} p(X=x)$$

or
$$\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \rightarrow \text{Markov process}$$

$\rightarrow$ transition matrix must have non-negative entries ("positivity") and columns summing to one ("completeness")

Analogously, we have for a quantum system:

- $\mathrm{Tr}[\mathcal{E}\rho] = 1$ (preservation of probability)
- $\mathcal{E}\left(\sum_i q_i \rho_i\right) = \sum_i q_i \mathcal{E}\rho_i$ (convex linear map)
- $(I_A \otimes \mathcal{E}_S)(\rho_{AS}) \geq 0$ for composite system AS

The time-evolution of a two-level system coupled with an environment is given by the masterequation:

$$\dot{\rho}(t) = -\frac{\gamma_+}{2}\left[\sigma_-\sigma_+\rho(t) + \rho(t)\sigma_-\sigma_+ - 2\sigma_+\rho(t)\sigma_-\right]$$

$$-\frac{\gamma_-}{2}\left[\sigma_+\sigma_-\rho(t) + \rho(t)\sigma_+\sigma_- - 2\sigma_-\rho(t)\sigma_+\right]$$

$$-\frac{\gamma_0}{2}\left[\sigma_z\rho(t) + \rho(t)\sigma_z - 2\sigma_z\rho(t)\sigma_z\right] = \mathcal{L}\rho(t)$$

where $\sigma_+ = |0\rangle\langle 1|$, $\sigma_- = |1\rangle\langle 0|$ and $\gamma_\alpha(\alpha = 0,+,-)$ are the decay rates of decay channels.

eigenoperators:

$$\mathcal{L}\sigma_1 = \frac{\gamma_+ + \gamma_- + 2\gamma_0}{2}\sigma_1 \equiv \lambda_1\sigma_1,$$

$$\mathcal{L}\sigma_2 = \frac{\gamma_+ + \gamma_- + 2\gamma_0}{2}\sigma_2 \equiv \lambda_2\sigma_2,$$

$$\mathcal{L}\sigma_3 = (\gamma_+ + \gamma_-)\sigma_3 \equiv \lambda_3\sigma_3,$$

$$\mathcal{L}\rho_{eq} = 0,$$

where $\rho_{eq} = \left(\gamma_+ |0\rangle\langle 0| + \gamma_- |1\rangle\langle 1|\right)/(\gamma_+ + \gamma_-)$

$\equiv (\sigma_0 + a\sigma_3)/2$ with

$a = (\gamma_+ - \gamma_-)/(2\gamma_+ + 2\gamma_-)$

The solution of the master eq is given by:

$$\mathcal{E}(t)\rho = p_0(t)\rho + \sum_{i=1,2,3} p_i(t)\sigma_i \rho \sigma_i$$

$$+ f(t)\left(\sigma_3\rho + \rho\sigma_3 - i\sigma_1\rho\sigma_2 + i\sigma_2\rho\sigma_1\right)$$

where

$$p_0(t) = \frac{1}{4}\left(1 + e^{-\lambda_1 t} + e^{-\lambda_2 t} + e^{-\lambda_3 t}\right)$$

$$p_1(t) = \frac{1}{4}\left(1 + e^{-\lambda_1 t} - e^{-\lambda_2 t} - e^{-\lambda_3 t}\right)$$

$$p_2(t) = \frac{1}{4}\left(1 - e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-\lambda_3 t}\right)$$

$$p_3(t) = \frac{1}{4}\left(1 - e^{-\lambda_1 t} - e^{-\lambda_2 t} + e^{-\lambda_3 t}\right)$$

$$f(t) = \frac{a}{4}\left(1 - e^{-\lambda_3 t}\right)$$

$\longrightarrow$ CPTP can be viewed as stochastic Pauli error with probabilities $p_i(t)$.

§2. Stabilizer Formalism and its applications

§2.1 Stabilizer Formalism

Define an n-qubit Pauli group $\mathcal{P}_n$:

$$\mathcal{P}_n := \{\pm 1, \pm i\} \times \{I, X, Y, Z\}^{\otimes n}$$

$\longrightarrow$ Pauli products

For example:

$$\mathcal{P}_2 := \{\pm 1, \pm i\} \times \{II, 1X, IY, IZ, \dots, ZZ\}$$

Next, define n-qubit stabilizer group $\mathcal{S}$ as an Abelian subgroup of $\mathcal{P}_n$:

$$\mathcal{S} := \{S_i\} \text{ s.t. } -I \notin \mathcal{S} \text{ and}$$
$$\forall\, S_i, S_j \in \mathcal{S} : [S_i, S_j] = 0$$

$\longrightarrow$ all elements are hermitian $S_i = S_i^\dagger$

define $\mathcal{S}_g$ as maximal independent subset (not expressible as prod of o. elem.)

Example 1:

$$\mathcal{S}_{Bell} = \{ II, XX, ZZ, -YY \}$$

$$(X \otimes X)(Z \otimes Z) = XZ \otimes XZ$$
$$= (-ZX) \otimes (-ZX)$$
$$= ZX \otimes ZX$$
$$= (Z \otimes Z)(X \otimes X)$$

We have $\mathcal{S}_{Bell} = \langle \{ XX, ZZ \} \rangle$ as

$$-Y \otimes Y = (X \otimes X)(Z \otimes Z)$$

Define "stabilizer state" as simultaneous eigenstate of all $S_i \in \mathcal{S}$ with eigenv. +1:

$$\forall\, S_i \in \mathcal{S} : S_i |\psi\rangle = |\psi\rangle$$

sufficient to demand:

$$\forall\, S_i \in \mathcal{S}_g : S_i |\psi\rangle = |\psi\rangle$$

Def $k := \# \mathcal{S}_g$ (number of elements)

→ divide Hilbert space into
2^k orthogonal subspaces

→ "stabilizer subspace": $2^d = 2^{n-k}$

thus for $n=k$, the quantum state is defined uniquely

Let us consider $\mathcal{S}_{Bell}$ again

→ stabilizer state of $\{XX, ZZ\}$ is
$(|00\rangle + |11\rangle)/\sqrt{2}$

stabilizer space of $\{ZZ\}$ is spanned by $|00\rangle$ and $|11\rangle$

→ Choosing $L_X = XX$ and $L_Z = ZI$, we can specify the state:

$$L_X (|00\rangle + |11\rangle)/\sqrt{2} = +1 (|00\rangle + |11\rangle)/\sqrt{2}$$

$$L_Z |00\rangle = +1 |00\rangle$$

Example 2:

Consider n-qubit state,

$$|cat\rangle = \frac{1}{\sqrt{n}} (|00\ldots 0\rangle + |11\ldots 1\rangle),$$

with stabilizer group spanned by

$$\left\langle Z_1 Z_2, \dots, Z_{n-1} Z_n, \prod_{i=1}^{n} X_i \right\rangle$$

removing $\prod_{i=1}^{n} X_i$ from the stabilizer generators, the stabilizer subspace becomes $\{|00\dots0\rangle, |11\dots1\rangle\}$

Choosing $L_x = \prod_{i=1}^{n} X_i$ and $L_z = Z_i$, we can access basis states of this subspace.